Magnetic Vector Potential ($\vec{A}$):—

the two vector identities for any scalar field V and vector field $\bar{A}$ are:-

$$\nabla \times (\nabla V) = 0 \quad \text{—— (1)}$$

$$\nabla \cdot (\vec{\nabla} \times \vec{A}) = 0 \quad \text{—— (2)}$$

Since, $E = -\nabla V$, Let us define the magnetic scalar potential V_m in terms of H as $\vec{H} = -\nabla V_m$ if $J = 0$ [∵ from Ampere's law $\vec{\nabla} \times \vec{H} = \vec{J}$]

—— ③

from Ampere's law, we have

$$\vec{\nabla} \times \vec{H} = \vec{J} \quad \text{—— (4)}$$

then by ③ & ④

$$\vec{J} = \vec{\nabla} \times (-\nabla V_m) = 0$$

or, $\boxed{\nabla^2 V_m = 0}$ (if $J = 0$)

—— ⑤

for a magnetostatics field,

$$\vec{\nabla}\cdot\vec{B}=0$$

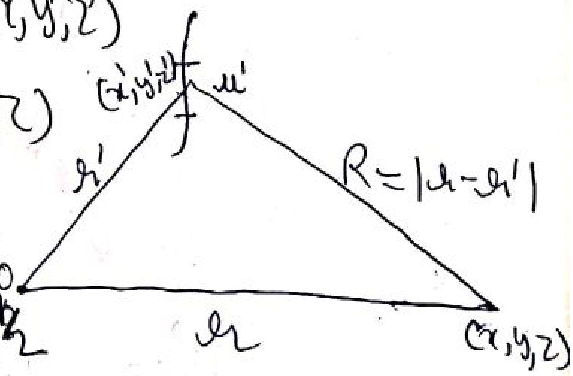
Let us define, vector magnetic potential $\underline{A}$ Such that

$$\vec{B}=\vec{\nabla}\times\vec{A} \quad \text{——— (6)}$$

according to, Bio Savart law
The magnetic fied due to a line current is

$$\vec{B}=\frac{\mu_0}{4\pi}\int_L \frac{I\,\vec{dl'}\times\vec{R}}{R^3} \quad \text{——— (7)}$$

R' is the distance vector from line element dl'. ~~to~~ the Source point (x',y',z') ~~to~~ the field point (x,y,z)
and,



$$R=|\vec{r}-\vec{r'}|$$

$$=[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}$$

$$\nabla\left(\frac{1}{R}\right)=-\frac{\vec{R}}{R^3}$$

or, $$\frac{\vec{R}}{R^3}=-\nabla\left(\frac{1}{R}\right)$$

or, $$\boxed{\frac{\vec{R}}{R^3}=\frac{\hat{a}_R}{R^2}} \quad \text{——— (8)}$$

This for direction

using (8) in (7), we will get,

$$\vec{B}=-\frac{\mu_0}{4\pi}\int_L I\,\vec{dl'}\times\nabla\left(\frac{1}{R}\right) \quad \text{——— (9)}$$

using Vector identity

$$\vec{\nabla} \times (f\vec{F}) = f(\vec{\nabla} \times \vec{F}) + (\vec{\nabla} \cdot f) \times \vec{F} \quad \text{——— (i)}$$

We have,

$$f = \frac{1}{R} \;\&\; \vec{F} = \vec{dl'}$$

then using (i), we get

$$\vec{dl'} \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}\vec{\nabla} \times \vec{dl'} - \left(\vec{\nabla} \times \frac{\vec{dl'}}{R}\right) \quad \text{——— (10)}$$

Since, $\vec{\nabla}$ operates w.r.to x, y, z and $\vec{dl'}$ is a function of x', y' & z' therefore,

$$\frac{1}{R}\vec{\nabla} \times \vec{dl'} = 0$$

then eqn (10) becomes,

$$\vec{dl'} \times \nabla\left(\frac{1}{R}\right) = -\vec{\nabla} \times \frac{\vec{dl'}}{R} \quad \text{——— (11)}$$

using eqn (11) in (9) we get,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \nabla \times \left(\frac{dl'}{R}\right)$$

$$\vec{B} = \vec{\nabla} \times \int_L \frac{\mu_0 I}{4\pi R} \vec{dl'} \quad \text{——— (12)}$$

on Comparing eqn (12) with eqn (6)

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We will get

$$\boxed{\vec{A} = \int_L \frac{\mu_0 I \vec{dl'}}{4\pi R}}$$

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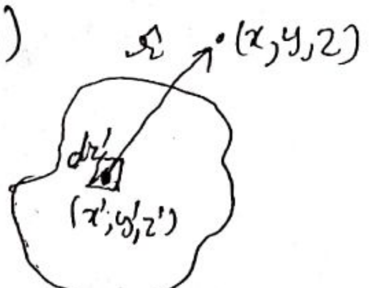
Divergence of $\vec{B}$:—

The Bio-Savart law for the general case of a volume current reads

$$\vec{B}(r) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau' \text{ ——— (i)}$$



This formula gives the magnetic field at a point $\vec{r} = (x, y, z)$ in terms of an integral over the current distribution $J(x', y', z')$

B is a function of (x, y, z)

J is a function of (x', y', z')

$$\hat{\mathscr{r}} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

$$d\tau' = dx'\, dy'\, dz'$$

Integration is over prime Co-ordinates and div. is taken w.r. to unprimed Co-ordinates.

Applying div. of (i)

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau' \text{ ——— (ii)}$$

Integrand is, (ii)

$$\because \vec{\nabla} \cdot \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot (\vec{\nabla} \times \vec{J}) - \vec{J} \cdot \left(\nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

But $\vec{\nabla} \times \vec{J} = 0$, bcoz $\vec{J}$ does not depend upon unprimed variables. (x, y, z), and $\because \vec{\nabla}\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)=0$

using above two relations, we get,

$$\vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)=0$$

Then by (ii),

$$\boxed{\vec{\nabla}\cdot\vec{B}=0}$$

the divergence of the magnetic field is zero

Curl of $\vec{B}$:— Applying the curl of eq^n (i)

$$\vec{\nabla}\times\vec{B}=\frac{\mu_0}{4\pi}\int\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' \qquad \text{(ii)}$$

In integrand using vector identity

$$\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)=\vec{J}\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)-(\vec{J}\cdot\vec{\nabla})\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \text{(iii)}$$

The second terms ~~was~~ integrates to zero,

S.t.

$$-(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}=(\vec{J}\cdot\nabla')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$(\vec{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right)=\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\cdot\vec{J}\right]-\left(\frac{x-x'}{\mathscr{r}^3}\right)(\underline{\nabla}'\cdot J)$$

$\because \underline{\nabla}'\cdot J=0$ (for study current)

$$(\vec{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right)=\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\cdot\vec{J}\right]$$

$$\left[-(\underline{J}\cdot\underline{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]=\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right]$$

and

$$\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)=4\pi\,\delta^3(\underline{\mathscr{r}})$$

put above relations in (iii)

$$\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)=\vec{J}\cdot4\pi\,\delta^3(\underline{\mathscr{r}})+\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right]$$